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On Sard's Quadrature Formulas of Order Two

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1. INTRODUCTION

Let an arbitrary weight function $w \in C[a, b]$ and nodes $y_{i,n}$ with

$$-\infty < a = y_{0,n} < y_{1,n} < \cdots < y_{n,n} = b < \infty \tag{1.1}$$

be given. We consider quadrature formulas (q.f.)

$$Q_n[f] = \sum_{i=0}^{n} a_{i,n} f(y_{i,n})$$

which are exact for polynomials of degree $\leq r-1$ and therefore admit a Peano kernel representation for the remainder $R_n[f]$ if $f^{(r-1)}$ is absolutely continuous; i.e.,

$$R_n[f] = \int_a^b f(x)\, w(x)\, dx - Q_n[f] = \int_a^b f^{(r)}(x)\, K_{r,n}(x)\, dx, \tag{1.2}$$

where $K_{r,n}(x) = R_n[(\cdot - x)_+^{r-1}/(r-1)!]$ is the Peano kernel of order r of Q_n. By (1.2),

$$|R_n[f]| \leq \|K_{r,n}\|_2 \, \|f^{(r)}\|_2 \qquad \text{for} \quad f \in W_2^r, \tag{1.3}$$

where $W_p^r = \{f \in C^{r-1}[a, b];\ f^{(r-1)}$ abs. cont., $\|f^{(r)}\|_p < \infty\}$, $\|f\|_p = (\int_a^b |f(x)|^p\, dx)^{1/p}$ $(1 \leq p < \infty)$, and $\|f\|_\infty = \|f\| = \operatorname{sup\,ess}_{a \leq x \leq b} |f(x)|$. A q.f. $Q_n = Q_n^s$ is called best in the sense of Sard (with respect to W_2^r, w, and $y_{0,n}, ..., y_{n,n}$), if it minimizes $\|K_{r,n}\|_2$, i.e., if it admits the least constant c in the estimate $|R_n[f]| \leq c\, \|f^{(r)}\|_2$.

The investigation of Sard's q.f. for integrals with a preassigned (integrable) weight function was suggested by Schoenberg in [5]. Schoenberg considered questions of existence and characterization for q.f. which

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contain also derivatives of f at the endpoints a and b of the interval of integration. In [3], Kershaw investigated Sard's q.f. of order two (i.e., $r=2$) for continuous weight functions. He obtained estimates of the L_2-norm of the corresponding Peano kernel $K^s_{2,n}$ and of the error $R^s_n[f]$ for $f \in C^4[a,b]$, but did not consider the weights $a^s_{i,n}$ (except for $w \equiv 1$ and equidistant nodes). Here, we will improve Kershaw's results on the norm of the Peano kernel and on the convergence of $R^s_n[f]$, and discuss the weights $a^s_{i,n}$ in more detail, especially if the nodes are given by certain node distribution functions z (i.e., $y_{i,n}=z(i/n)$). We propose a generalization of the first and second conjecture of Meyers and Sard [4], which holds in the case considered here.

2. Arbitrary Nodes

From now on, we restrict to the case $r=2$. For the nodes $y_i = y_{i,n}$, only (1.1) is supposed to hold, and w is the preassigned continuous weight function. Let Q^s_n be Sard's q.f. of order two, R^s_n the corresponding remainder functional, and K^s_n the corresponding Peano kernel of order two. Further let

$$h_j = y_j - y_{j-1} \qquad \text{and} \qquad \Delta_n = \max\{h_j;\, j=1, ..., n\}.$$

THEOREM 2.1. (a) $\|K^s_n\|_2 \leqq ((b-a)/120)^{1/2}\, \Delta_n^2\, \|w\|$, *and*

(b) $\|K^s_n\| \leqq 3\Delta_n^2\, \|w\|/8$.

Remark. Theorem 2.1(a) was proven by Kershaw [3] with the constant $(3/64)^{1/2} = 0.216506...$, whereas $(1/120)^{1/2} = 0.091287....$ This constant is best possible since, for $n=1$ and $w=\text{constant}$, we have equality in (a). From Theorem 2.1(a) and (1.3), we get the following

COROLLARY. $|R^s_n[f]| \leqq ((b-a)/120)^{1/2}\, \Delta_n^2\, \|w\|\, \|f''\|_2$ *for* $f \in W^2_2$.

If f'' is smooth, better estimates can be obtained. Let $\omega(f, t)$ denote the modulus of continuity of f, i.e., $\omega(f,t) = \sup\{|f(x)-f(y)|;\, |x-y| \leqq t\}$.

THEOREM 2.2. *Let* $r_n[f] = 3\Delta_n^3\, \|w\|\, (|f''(a)| + |f''(b)|)/16$. *Then*

(a) $|R^s_n[f]| \leqq (b-a)\Delta_n^2 \omega(f'', \Delta_n)\, \|w\|/120^{1/2} + r_n[f]$ *for* $f \in C^2[a,b]$,

(b) $|R^s_n[f]| \leqq (b-a)\, \Delta_n^3 \omega(f''', \Delta_n)\, \|w\|/60 + r_n[f]$ *for* $f \in C^3[a,b]$,

(c) $|R^s_n[f]| \leqq (b-a)\, \Delta_n^4\, \|f^{(4)}\|\, \|w\|/120 + r_n[f]$ *for* $f \in C^4[a,b]$.

Remarks. (a) Kershaw [3] proved $R^s_n[f] = O(\Delta_n^{5/2})$ for $f \in C^4[a,b]$, whereas Theorem 2.2(b) gives $R^s_n[f] = O(\Delta_n^3)$ even for $f \in C^3[a,b]$. This result cannot be improved further, since for $w \equiv 1$, equidistant nodes (i.e., $y_i = a + i\Delta_n$, $\Delta_n = (b-a)/n$), and $f \in W_1^4$, we have the following asymptotic behaviour of $R_n^s[f]$ (the proof will be omitted):

$$R_n^s[f] = -\Delta_n^3(f''(a) + f''(b))3^{1/2}/72 + \Delta_n^4(f'''(b) - f'''(a))/720 + o(\Delta_n^4).$$

(b) For $f \in C^4[a, b]$ with $f''(a) = f''(b) = 0$, Theorem 2.2(c) gives $|R_n^s[f]| \leqq (b-a)\,\Delta_n^4\,\|f^{(4)}\|\,\|w\|/120$. This was also proven by Kershaw [3] with the constant $3^{3/2}/64 = 0.081189...$ $(1/120 = 0.008333...)$.

(c) At least for $w \equiv 1$ and equidistant nodes, the constants in Theorem 2.2 are not best possible. Schurer [7] has proven that for $w \equiv 1$, equidistant nodes and $f \in C^4[a, b]$,

$$|R_n^s[f]| \leqq \Delta_n^3\,|f''(a) + f''(b)|/40 + (b-a)\,\Delta_n^4\,\|f^{(4)}\|/320.$$

where the constants 1/40 and 1/320 are best possible.

For the weights $a_{i,n}^s$ of Q_n^s, the following estimate in dependence of the global mesh ratio holds.

THEOREM 2.3. *Let $h_i/h_j \leqq M$ for all $i, j = 1, ..., n$. Then*

$$|a_{i,n}^s| \leqq \Delta_n\,\|w\|\,(1+M) \qquad \text{for} \quad i = 0, ..., n.$$

3. SPECIAL NODE DISTRIBUTIONS

We now consider the case that the nodes are given by a node distribution function z, i.e.,

$$y_{i,n} = z(x_{i,n}), \qquad \text{where} \quad x_{i,n} = x_i = ih,\ h = 1/n.$$

For simplicity, only the following two classes of node distributions will be considered:

$$Z_1 = \{z \in C^1[0, 1];\ z(0) = a,\quad z(1) = b,\quad z'(x) > 0 \text{ for } 0 \leqq x \leqq 1\},$$

$$Z_2 = \{z \in C^2[0, 1];\ z(0) = a,\quad z(1) = b,\quad z'(x) > 0 \text{ for } 0 < x < 1,$$
$$z'(0) = z'(1) = 0,\quad z''(0) \neq 0,\quad z''(1) \neq 1\}.$$

Important examples are (i) for $z \in Z_1$: $z(x) = a + x(b-a)$ (equidistant nodes), and (ii) for $z \in Z_2$: $z(x) = -\cos \pi x$ with $a = -1$, $b = 1$ (nodes of the Clenshaw–Curtis q.f.). Let

$$\lambda = 3^{1/2} - 2 \qquad \text{and} \qquad \lambda_i = \lambda_{i,n} = (\lambda^i + \lambda^{n-i})/(1 + \lambda^n). \tag{3.1}$$

Further, for $z \in C^1[0, 1]$, let

$$a^0_{i,n} = hz'(x_i)\, w(y_i)(5 + \lambda_{1,n})/12, \qquad i = 0, n,$$
$$a^0_{i,n} = hz'(x_i)\, w(y_i)(1 - \lambda_{i,n}/2), \qquad i = 1, ..., n-1. \tag{3.2}$$

THEOREM 3.1. *Let $z \in Z_j$, $j = 1, 2$, and $a^s_{i,n}$ the weights of Q^s_n.*

(a) *There exists a constant $d = d(w, z)$ (i.e., d depends on w and z only) with $|a^s_{i,n}| \leq d/n$ for all i and n.*

(b) *$a^s_{i,n} = hz'(x_i)\, w(y_i) + o(h)$ if $\varepsilon \leq x_i \leq 1 - \varepsilon$, $0 < \varepsilon < \frac{1}{2}$, and the o-term holds uniformly in i. If $z \in Z_1$ only, then*

$$a^s_{i,n} = a^0_{i,n} + o(h) \qquad \text{uniformly for all} \quad i = 0, ..., n.$$

(c) *If $i = i(n)$ depends on n such that $\lim_{n \to \infty} i(n)/n = x \in (0, 1)$, then $\lim_{n \to \infty} na^s_{i(n),n} = z'(x)\, w(z(x))$.*

(d) *$\lim_{n \to \infty} na^s_{i,n} = z'(0)\, w(a)(1 - \lambda^i/2)$ for any fixed $i \geq 1$, and $\lim_{n \to \infty} na^s_{0,n} = z'(0)\, w(a)(5 + \lambda)/12$.*

(Corresponding results are obtained if $n - i$ is fixed.)

For the best q.f. with respect to W^r_2 $(r \geq 1)$, $w \equiv 1$, and $z(x) = x$, three conjectures were set up by Meyers and Sard [4], which were proven by Schoenberg in [6]. The first two conjectures, which concern the weights (the third concerns the L_2-norm of the corresponding Peano kernel and will not be considered here) are as follows:

$$\text{(MS1)} \quad \lim_{n \to \infty} na^s_{[n/2]+i,n} = 1 \qquad \text{for any fixed integer } i,$$

and

$$\text{(MS2)} \quad \lim_{n \to \infty} na^s_{i,n} \qquad \text{exists for any fixed integer } i \geq 0$$

($[x]$ denotes the largest integer not greater than x). Theorem 3.1 suggests the following generalization of these conjectures for the best q.f. with respect to W^r_2 $(r \geq 1)$, $w \in C[a, b]$, and $y_i = z(x_i)$, $z \in C^1[0, 1]$ strictly increasing:

$$\text{(GMS1)} \quad \lim_{n \to \infty} na^s_{i(n),n} = z'(x)\, w(z(x)) \qquad \text{if} \quad \lim_{n \to \infty} i(n)/n = x \in (0, 1),$$

and

$$\text{(GMS2)} \quad \lim_{n \to \infty} na^s_{i,n} \qquad \text{exists for any fixed integer } i \geq 0.$$

We return to the case $r=2$ and conclude with the following theorem, which is an easy consequence of Theorem 3.1(a) and (b) (the proof will be omitted).

THEOREM 3.2. *Let $z \in Z_j$, $j=1, 2$. Then the following is valid.*

(a) $\lim_{n \to \infty} \sum_{i=0}^{n} |a_{i,n}^s| = \int_a^b |w(x)|\, dx$,

(b) $\lim_{n \to \infty} Q_n^s[f] = \int_a^b f(x)\, w(x)\, dx$ *if f is Riemann integrable.*

4. The Proofs of Section 2

Let $g \in C^2[a, b]$ be any fixed function with

$$g'' = w,$$

and let s_j be the B-splines of degree 1, i.e., for $j=0, ..., n$

$$s_j(x) = \begin{cases} (x - y_{j-1})/h_j & \text{for} \quad x \in (y_{j-1}, y_j) \\ (y_{j+1} - x)/h_{j+1} & \text{for} \quad x \in [y_j, j_{j+1}) \\ 0 & \text{else} \end{cases}$$

($y_{-1} < a$ and $y_{n+1} > b$ may be chosen arbitrary). Now the set of all Peano kernels $K_n = K_{2,n}$ of order two of q.f. with nodes $y_0, ..., y_n$ is given by

$$G_n = \left\{ K_n = g + \sum_{i=0}^{n} c_i s_i;\, c_i \in \mathbb{R},\, K_n(a) = K_n(b) = 0 \right\}.$$

Since G_1 consists of one element only (viz. $L_1[g]$, s. (4.4)), we assume that $n \geqq 2$. For any fixed $K_n^* \in G_n$, we get

$$G_n = \left\{ K_n = K_n^* + \sum_{i=1}^{n-1} c_i s_i;\, c_i \in \mathbb{R} \right\}.$$

Therefore, a q.f. Q_n^s is best in the sense of Sard (with respect to W_2^2, w, and $y_0, ..., y_n$), if its second Peano kernel K_n^s satisfies

$$\|K_n^s\|_2 = \min_{c_1, ..., c_{n-1}} \left\| K_n^* + \sum_{i=1}^{n-1} c_i s_i \right\|_2, \tag{4.1}$$

and, as a consequence, K_n^s is also determined by

$$\int_a^b s_i(x)\, K_n^s(x)\, dx = 0 \qquad \text{for} \quad i = 1, ..., n-1. \tag{4.2}$$

If $K_n^s = K_n^* + \sum_{i=1}^{n-1} c_i^s s_i$, then the c_i^s's are the solution of

$$c_{i-1} h_i/(h_i + h_{i+1}) + 2c_i + c_{i+1} h_{i+1}/(h_i + h_{i+1}) \\ = r_i, \qquad i = 1, ..., n-1, \tag{4.3}$$

with $c_0 = c_n = 0$ and $r_i = -6 \int_a^b s_i(x) K_n^*(x)\, dx/(h_i + h_{i+1})$. System (4.3) is well known, e.g., from cubic spline interpolation.

LEMMA 4.1. *The solution of* (4.3) *satisfies*

(a) $\max\{|c_i|;\, i = 1, ..., n-1\} \leqq \max\{|r_i|;\, i = 1, ..., n-1\}$ *and*

(b) $|c_i| \leqq \sum_{j=1}^{n-1} |r_j|\, 2^{1-|i-j|}/3$, $i = 1, ..., n-1$.

For (a) s. de Boor [1, p. 43/44], for (b) see Kershaw [2]. In the sequel we will make essential use of linear spline interpolation. Let $L_n[f]$ be the corresponding error, i.e.,

$$L_n[f](x) = f(x) - \sum_{i=0}^{n} f(y_i)\, s_i(x), \qquad x \in [a, b]. \tag{4.4}$$

If $x \in [y_i, y_{i+1}]$, then $L_n[f](x) = (x - y_i)(x - y_{i+1})[y_i, y_{i+1}, x] f$ (de Boor [1, p. 39]; $[\cdot,\cdot,\cdot]$ denotes the second divided diffence), and from this it is easy to get the following lemma.

LEMMA 4.2. (a) *Let* $f \in C^2[a, b]$. *Then, for* $j = 1, ..., n-1$, $\int_a^b s_j(x) L_n[f](x)\, dx = -f''(\eta_j)(h_j^3 + h_{j+1}^3)/24$, *where* $\eta_j \in [y_{j-1}, y_{j+1}]$.

(b) $\|L_n[f]\|_2 \leqq ((b-a)/120)^{1/2} \Delta_n^2 \|f''\|$ *and* $\|L_n[f]\| \leqq \Delta_n^2 \|f''\|/8$ *for* $f \in C^2[a, b]$.

(c) $\|L_n[f]\|_2 \leqq ((b-a)/30)^{1/2} \Delta_n \omega(f', \Delta_n)$ *and* $\|L_n[f]\| \leqq \Delta_n \omega(f', \Delta_n)/4$ *for* $f \in C^1[a, b]$.

(d) $\|L_n[f]\|_2 \leqq (b-a)^{1/2} \omega(f, \Delta_n)$ *and* $\|L_n[f]\| \leqq \omega(f, \Delta_n)$ *for* $f \in C[a, b]$.

Obviously, $L_n[g] \in G_n$. $L_n[g]$ is the Peano kernel of the (generalized) trapezoidal rule.

LEMMA 4.3. *Let* $K_n^s = L_n[g] + \sum_{i=1}^{n-1} c_i s_i$. *Then* $|c_i| \leqq \Delta_n^2 \|w\|/4$, $i = 1, ..., n-1$.

Proof. From (4.3) with $K_n^* = L_n[g]$ and Lemma 4.2(a) we get

$$|r_j| = (h_j + h_{j+1})^{-1} (h_j^3 + h_{j+1}^3)\, |w(\eta_j)|/4 \\ \leqq \max(h_j^2, h_{j+1}^2)\, \|w\|/4 \leqq \Delta_n^2 \|w\|/4. \tag{4.5}$$

An application of Lemma 4.1(a) completes the proof. Q.E.D.

Proof of Theorem 2.1. (a) Since $L_n[g] \in G_n$, (4.1) and Lemma 4.2(b) give

$$\|K_n^s\|_2 \leqq \|L_n[g]\|_2 \leqq ((b-a)/120)^{1/2} \Delta_n^2 \|w\|.$$

(b) From Lemma 4.3 and Lemma 4.2(b), we get

$$\|K_n^s\| \leqq \|L_n[g]\| + \left\|\sum s_i\right\| \max |c_i| \leqq (\tfrac{1}{8} + \tfrac{1}{4}) \Delta_n^2 \|w\|. \qquad \text{Q.E.D.}$$

Proof of Theorem 2.2. By (1.2) and (4.2) we have

$$\begin{aligned} R_n^s[f] &= \int_a^b \left(f''(x) - \sum_{i=1}^{n-1} f''(y_i)\, s_i(x)\right) K_n^s(x)\, dx \\ &= \int_a^b L_n[f''](x)\, K_n^s(x)\, dx \\ &\quad + \int_a^b f''(y_0)\, s_0(x)\, K_n^s(x)\, dx \\ &\quad + \int_a^b f''(y_n)\, s_n(x)\, K_n^s(x)\, dx, \end{aligned}$$

which gives

$$\begin{aligned} |R_n^s[f]| &\leqq \|L_n[f'']\|_2\, \|K_n^s\|_2 \\ &\quad + (|f''(a)| + |f''(b)|)\, \|K_n^s\|\, \Delta_n/2. \end{aligned}$$

Now everything follows directly from Lemma 4.2 and Theorem 2.1. Q.E.D.

By Schoenberg [5], we have $a_{0,n} = -K_n'(a+)$, $a_{n,n} = K_n'(b-)$, and $a_{i,n} = K_n'(y_i-) - K_n'(y_i+)$, $i = 1, ..., n-1$. Some simple computations give

LEMMA 4.4. *Let* $K_n = L_n[g] + \sum_{i=1}^n c_i s_i$, $c_0 = c_n = 0$, *and let* $a_{i,n}$ *be the weights of the corresponding q.f. Then*

$$a_{0,n} = h_1 w(\zeta_0)/2 - c_1/h_1, \qquad a_{n,n} = h_n w(\zeta_n)/2 - c_{n-1}/h_n$$

and

$$\begin{aligned} a_{i,n} &= (h_i + h_{i+1})\, w(\zeta_i)/2 + (c_i - c_{i+1})/h_{i+1} \\ &\quad + (c_i - c_{i-1})/h_i, \qquad i = 1, ..., n-1, \end{aligned}$$

where $\zeta_0 \in [y_0, y_1]$, $\zeta_n \in [y_{n-1}, y_n]$, *and* $\zeta_i \in [y_{i-1}, y_{i+1}]$, $i = 1, ..., n-1$.

Theorem 2.3 is an immediate consequence of Lemma 4.3 and Lemma 4.4.

5. The Proof of Theorem 3.1

Let $\lambda = 3^{1/2} - 2$ and $\lambda_i = (\lambda^i + \lambda^{n-i})/(1+\lambda^n)$ as in (3.1). Then

$$\lambda \in (-1, 0), \qquad 1 + 4\lambda + \lambda^2 = 0, \qquad |\lambda_i| \leqq 1, \qquad i = 0, ..., n. \tag{5.1}$$

Lemma 5.1. *Let* $K_n^s = L_n[g] + \sum_{i=1}^{n-1} (w(y_i)\, h_i h_{i+1}(1-\lambda_i)/12 + \varepsilon_i) s_i$ *and* $z \in Z_j$, $j = 1, 2$. *Then there exists a constant* $d = d(w, z)$ *such that*

$$|\varepsilon_i| \leqq dh^2(\omega(w, h) + \omega(z', h)) \qquad \textit{for} \quad i = 1, ..., n-1.$$

Proof. Lemma 4.1(a) gives $\max |\varepsilon_i| \leqq \max |r_i|$ with

$$r_i = -6 \int_a^b s_i(x) \left(L_n[g](x) + \sum_{j=0}^{n} w(y_j) \times h_j h_{j+1}(1-\lambda_j)\, s_j(x)/12 \right) dx \Big/ (h_i + h_{i+1})$$

(note that $1 - \lambda_0 = 1 - \lambda_n = 0$; further let $h_0 = h_1$ and $h_{n+1} = h_n$). Lemma 4.2(a) and some elementary calculations give $12(h_i + h_{i+1}) r_i = w(\eta_i) A + B + C$, where

$$A = h_i^2(2(h_i - h_{i+1}) + h_i - h_{i-1}) + h_{i+1}^2(2(h_{i+1} - h_i) + h_{i+1} - h_{i+2}),$$

$$B = w(y_{i-1})\, h_{i-1} h_i^2 \lambda_{i-1} + 2w(y_i)\, h_i h_{i+1}(h_i + h_{i+1}) \lambda_i + w(y_{i+1})\, h_{i+1}^2 h_{i+2} \lambda_{i+1}$$

$$C = (w(\eta_i) - w(y_{i-1}))\, h_{i-1} h_i^2 + 2(w(\eta_i) - w(y_i))\, h_i h_{i+1}(h_i + h_{i+1}) + (w(\eta_i) - w(y_{i+1}))\, h_{i+1}^2 h_{i+2}.$$

(i) $|h_{j+m} - h_j| = |\int_{x_{j+m-1}}^{x_{j+m}} (z'(x) - z'(x - mh))\, dx| \leqq mh\omega(z', h)$, which gives $|A| \leqq 3\Delta_n h\omega(z', h)(h_i + h_{i+1})$.

(ii) Statement (5.1) gives $\lambda_i = -(\lambda_{i-1} + \lambda_{i+1})/4$. Inserting this in B gives, together with $|\lambda_i| \leqq 1$,

$$|B| \leqq |w(y_{i-1})\, h_{i-1} h_i^2 - w(y_i)\, h_i h_{i+1}(h_i + h_{i+1})/2| + |w(y_{i+1})\, h_{i+1}^2 h_{i+2} - w(y_i)\, h_i h_{i+1}(h_i + h_{i+1})/2| = B_1 + B_2.$$

The triangle inequality gives

$$\begin{aligned} B_1 &\leqq \omega(w, \Delta_n)\, h_{i-1} h_i^2 + \|w\|\, h_i\, |h_{i-1} h_i - h_{i+1}(h_i + h_{i+1})/2| \\ &\leqq e_1 h^2(\omega(w, h) + \omega(z', h)) h_i. \end{aligned}$$

This is also true for B_2 with the same constant $e_1 = e_1(w, z)$ and h_i replaced by h_{i+1}.

(iii) $|C| \leqq e_2 h^2 \omega(w, h)(h_i + h_{i+1})$. Q.E.D.

LEMMA 5.2. *Let $K_n^s = L_n[g] + \sum_{i=1}^{n-1} c_i s_i$ and $z \in Z_2$. Then*

$$|c_i| \leqq d \min(i^2, (n-i)^2) h^4 \qquad \textit{for} \quad i = 1, ..., n-1, \textit{ where } d = d(w, z).$$

Proof. (i) $h_i = hz'(\xi_i) = h(z'(\xi_i) - z'(0))$, $\xi_i \in [x_{i-1}, x_i]$ gives

$$h_i \leqq ih^2\, \|z''\|, \qquad i = 1, ..., n. \tag{5.2}$$

(ii) From (4.5) and (5.2) we get $|r_j| \leqq \|w\|\, (j+1)^2\, h^4\, \|z''\|^2/4$ and therefore, by Lemma 4.1(b),

$$\begin{aligned} |c_i| &\leqq e_1 h^4 \sum_{j=1}^{n-1} (j+1)^2\, 2^{-|j-i|} \\ &= e_1 i^2 h^4 \sum_{j=1}^{n-1} ((j+1)/i)^2\, 2^{-|j-i|} \leqq e_2 i^2 h^4, \end{aligned}$$

where $e_j = e_j(w, z)$, $j = 1, 2$. The other estimates follow similar. Q.E.D.

We are now ready to prove the following lemma on $a_{i,n}^s$, from which Theorem 3.1 follows immediately.

LEMMA 5.3. (a) *Let $z \in Z_1$. Then there exist constants $d_j = d_j(w, z)$, $j = 1, 2$, with*

$$|a_{i,n}^s - a_{i,n}^0| \leqq d_1 h(\omega(z', h) + \omega(w, h))$$

and

$$|a_{i,n}^s| \leqq d_2 h \qquad \textit{for} \quad i = 0, ..., n.$$

(b) *Let $z \in Z_2$ and $\varepsilon \in (0, \frac{1}{2})$. Then the estimates of* (a) *hold for*

$$\varepsilon + 1/n \leqq i/n \leqq 1 - \varepsilon - 1/n \qquad \textit{with} \quad d_j = d_j(w, z, \varepsilon),\ j = 1, 2.$$

(c) *Let $z \in Z_2$ and $\varepsilon \in (0, \frac{1}{2})$. Then, for ε sufficiently small, there exists a constant $d_3 = d_3(w, z, \varepsilon)$ such that*

$$|a^s_{i,n}| \leqq d_3 h^2 \min(i+1, n-i+1) \qquad \text{if} \quad 0 \leqq i/n \leqq \varepsilon - 1/n$$
$$\text{or} \quad 1 - \varepsilon + 1/n \leqq i/n \leqq 1.$$

Proof. (a) Let K^s_n be as in Lemma 5.1 and $z \in Z_1$. For $i = 1, ..., n-1$, Lemma 4.4 gives $a^s_{i,n} = a^1_i + a^2_i + a^3_i$ with $a^1_i = (h_i + h_{i+1})\, w(\zeta_i)/2$, $\zeta_i \in [y_{i-1}, y_{i+1}]$, $a^2_i = -(w(y_{i-1})\, h_{i-1}(1 - \lambda_{i-1}) - w(y_i)\ (h_i + h_{i+1})(1 - \lambda_i) + w(y_{i+1})\, h_{i+2}(1 - \lambda_{i+1}))/12$, and $a^3_i = (\varepsilon_i - \varepsilon_{i+1})/h_{i+1} + (\varepsilon_i - \varepsilon_{i-1})/h_i$.

(i) $|a^1_i - hz'(x_i)\, w(y_i)| = h\ |z'(\xi_i)\, w(\zeta_i) - z'(x_i)\, w(y_i)|$, $\xi_i \in [x_{i-1}, x_{i-1}]$. The triangle inequality gives $|a^1_i - hz'(x_i)\, w(y_i)| \leqq e_1 h(\omega(z', h) + \omega(w, h))$.

(ii)

$$\begin{aligned}
|a^2_i + hz'(x_i)\, w(y_i)\lambda_i/2| = {} & |w(y_i)(h_i + h_{i+1})(\lambda_{i-1} - 2\lambda_i + \lambda_{i+1})/2 \\
& - (w(y_{i-1})h_{i-1} - w(y_i)(h_i + h_{i+1})/2)(1 - \lambda_{i-1}) \\
& - (w(y_{i+1})h_{i+2} - w(y_i)(h_i + h_{i+1})/2) \\
& \times (1 - \lambda_{i+1}) + 6hz'(x_i)\, w(y_i)\lambda_i|/12 \\
\leqq {} & e_2 h(\omega(z', h) + \omega(w, h)),
\end{aligned}$$

since (5.1) gives $\lambda_{i-1} - 2\lambda_i + \lambda_{i+1} = -6\lambda_i$.

(iii) By Lemma 5.1 and $h_i = z'(\xi_i)h \geqq h \min z'(x) > 0$, we get

$$\begin{aligned}
|a^3_i| &\leqq dh^2(\omega(z', h) + \omega(w, h))(2/h_i + 2/h_{i+1}) \\
&\leqq e_3 h(\omega(z', h) + \omega(w, h)).
\end{aligned}$$

Combining (i)–(iii) gives the first statement of Lemma 5.3(a) for $i = 1, ..., n-1$. The case $i = 0$, n can be treated in the same way. The second statement follows directly from Theorem 2.3.

(b) follows by minor modifications of the proof of (a).

(c) For $z \in Z_2$ and ε sufficiently small, we have $z_\varepsilon := \min\{|z''(x)|;\ 0 \leqq x \leqq \varepsilon \text{ or } 1 - \varepsilon \leqq x \leqq 1\} > 0$. If $0 < j/n \leqq \varepsilon$, then

$$h_j = \int_{x_{j-1}}^{x_j} \int_0^x z''(t)\, dt\, dx \geqq z_\varepsilon h^2(j - 1/2) > 0. \tag{5.3}$$

Let K^s_n be as in Lemma 5.2. Then Lemma 4.4 and (5.2) give (for $i = 1, ..., n-1$)

$$|a^s_{i,n}| \leqq e_1(i+1)h^2 + e_2(i+1)^2\, h^4(1/h_i + 1/h_{i+1}) \leqq d_3(i+1)h^2$$

by (5.3). The other cases can be treated similarly. Q.E.D.

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